Exercise 5

Verify that $\sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk}\alpha_{jk} = 0$ if $\alpha_{jk} = \alpha_{kj}$.

Solution

ε_{ijk} is the permutation symbol, which is defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, \text{ or } 312 \\ -1 & \text{if } ijk = 321, 132, \text{ or } 213 \\ 0 & \text{if any indices are the same} \end{cases}.$$

Evaluating the double sum, we have

$$\sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk}\alpha_{jk} = \sum_{j=1}^3 (\varepsilon_{ij1}\alpha_{j1} + \varepsilon_{ij2}\alpha_{j2} + \varepsilon_{ij3}\alpha_{j3}) = \sum_{j=1}^3 \varepsilon_{ij1}\alpha_{j1} + \sum_{j=1}^3 \varepsilon_{ij2}\alpha_{j2} + \sum_{j=1}^3 \varepsilon_{ij3}\alpha_{j3}$$

Expand each of the sums. The permutation symbol is 0 when two of its indices are the same.

$$\begin{aligned} &= \underbrace{\varepsilon_{i11}\alpha_{11}}_{=0} + \varepsilon_{i21}\alpha_{21} + \varepsilon_{i31}\alpha_{31} \\ &\quad + \varepsilon_{i12}\alpha_{12} + \underbrace{\varepsilon_{i22}\alpha_{22}}_{=0} + \varepsilon_{i32}\alpha_{32} \\ &\quad + \varepsilon_{i13}\alpha_{13} + \varepsilon_{i23}\alpha_{23} + \underbrace{\varepsilon_{i33}\alpha_{33}}_{=0} \end{aligned}$$

Switch the indices of ε like so. Doing this results in a minus sign.

$$\begin{aligned} &= \underbrace{\varepsilon_{i11}\alpha_{11}}_{=0} - \varepsilon_{i12}\alpha_{21} - \varepsilon_{i13}\alpha_{31} \\ &\quad + \varepsilon_{i12}\alpha_{12} + \underbrace{\varepsilon_{i22}\alpha_{22}}_{=0} - \varepsilon_{i23}\alpha_{32} \\ &\quad + \varepsilon_{i13}\alpha_{13} + \varepsilon_{i23}\alpha_{23} + \underbrace{\varepsilon_{i33}\alpha_{33}}_{=0} \end{aligned}$$

Use the fact that $\alpha_{jk} = \alpha_{kj}$ to switch the indices of α. Consequently, every term cancels.

$$\begin{aligned}
&= \underbrace{\varepsilon_{i11}\alpha_{11}}_{=0} - \cancel{\varepsilon_{i12}\alpha_{12}} - \cancel{\varepsilon_{i13}\alpha_{13}} \\
&\quad + \cancel{\varepsilon_{i12}\alpha_{12}} + \underbrace{\varepsilon_{i22}\alpha_{22}}_{=0} - \cancel{\varepsilon_{i23}\alpha_{23}} \\
&\quad + \cancel{\varepsilon_{i13}\alpha_{13}} + \cancel{\varepsilon_{i23}\alpha_{23}} + \underbrace{\varepsilon_{i33}\alpha_{33}}_{=0} \\
&= 0
\end{aligned}$$

Therefore, if $\alpha_{jk} = \alpha_{kj}$, then

$$\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\alpha_{jk} = 0.$$